Projection and Assembly Method for Multibody Component Model Reduction

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The problem addressed is that of obtaining reduced-order component models for use in simulating the dynamics of a multibody system. In certain cases, nonlinear system models may be constructed using linear dynamic models for each component, but allowing large angle motion between components. Without some form of model reduction, system models constructed in this manner may be too large for use in control system design and simulation trades. This paper analyzes one method of component model reduction that allows system level requirements (e.g., capturing the effect of body 1 reaction wheel noise on body 2 camera pointing) to aid in the selection of the reduced-order component models. Briefly stated, important modes are selected at the system level and projected onto the components, and reduced-order components are then assembled into a reduced-order system model that retains the projected modes.

Introduction

THE problem to be solved is that of simulating the dynamics of a multibody system. A multibody system comprises two or more bodies or components connected at hinges. In general, the bodies may be rigid or flexible, and the hinges may have from one to six independent degrees of freedom. Often all deformations of each body from its reference condition are in the linear range, whereas the resulting system dynamics is nonlinear. In this case, nonlinear system models may be constructed using linear dynamic models for each component, but allowing large angle motion between components. This is the approach used in a number of existing multibody software tools.

The problem is that system models constructed in this manner may be too large for use in control system design and simulation trades. Model reduction is needed to bring the model down to manageable size. If the system model is available in linear form, system model reduction can be applied directly. For the class of multibody problems just discussed, only the component models are available in linear form, and existing multibody software can be used if we reduce the component models before assembly into the system model. A multibody system is inherently a geometrically nonlinear system because of the time-varying, large angle articulation between bodies.

Component model reduction is typically done to some level anyway if the source of the model is a finite-element program. This first level of model reduction often uses some simple criterion such as, "keep all cantilever modes below 40 Hz." The challenge is to reduce the component model further in some manner that preserves how the component behaves when connected to the complete system and how the component affects system level requirements. The projection and assembly method described in this paper attempts to do this.

Model reduction for linear systems has been addressed by a number of researchers, resulting in a variety of suggested linear system model reduction methods.[1-4] Less attention has been paid to the problem of model reduction for components of multibody systems. Component modal synthesis methods[5-8] have the capability of producing reduced-order component models, but typically do so based on component-level rather than system-level criteria. When only one body in a multibody system is flexible, Macala[9] captures desired system modes exactly by augmenting the flexible body by the mass and inertia of the rigid body. A subset of the free-free modes of this augmented body are then used as the flexible body component modes. Eke and Man[10] extend this capability to systems of more than one flexible body with a method that involves choosing system modes of interest, projecting the mode shapes of these desired modes onto each flexible component, reducing the order of each component accordingly, and assembling the components into a system model. Upon assembly, each of the original desired system modes is recovered exactly (to the numerical precision of the algorithm).

This paper analyzes the method outlined in Ref. 10 to show why the desired modes are returned exactly, presents necessary conditions for the success of the procedure, and proposes an extension to the method to handle situations when these necessary conditions are not met. Simple examples are presented to demonstrate the workings of the algorithm. The name "projection and assembly method" is used to describe this component model-reduction method.

Description of Method

The idea of the projection and assembly method is to decide what system modes are important and to choose component models that, when assembled, capture those important system modes. The projection and assembly method is described in detail in Ref. 10. It works as follows:

1) Acquire component models.

2) Synthesize the system model in some configuration of interest.

3) Apply any system-level model reduction desired to choose which system free-free modes to retain.

4) Project the mode shapes of these retained modes onto each component.

5) Choose new component states such that only these projected modes are admissible motions.

6) Transform the component models into reduced-order component models using these new states.

7) Assemble the reduced-order component models into a reduced-order system model.

One side effect of the method is apparent by doing a little arithmetic. Assume we have a three-flexible-body system with bodies 2 and 3 connected to body 1 by simple one-degree-of-

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freedom (DOF) hinges. If each body has 100 free-free DOF, then the system has 290 DOF (100 + 100 + 100 − 5 − 5; each single DOF hinge corresponds to five constraints). Assume that through a system model-reduction analysis 20 important modes are found. The projection and assembly method will project these 20 modes onto *each* body. When the reduced-order component models are assembled, the reduced order system model will have 50 modes (20 + 20 + 20 − 5 − 5). These 50 include the 20 desired modes plus 30 "extraneous modes."

Why does this method work? Briefly, since each component's portion of the mode shapes of interest is explicitly retained as admissible motions for each component, and since these modes of the complete system automatically satisfy all constraints, the desired modes will again be admissible motions when the system is assembled. That they will also be modes of the reduced-order system is shown in the following analysis.

Analysis

Component Equations of Motion

Assume we have n_b bodies or components. The unconstrained equations of motion of each may be expressed as

$$M_i\ddot{x}_i + K_i x_i = G_i u, \qquad i = 1,\ldots, n_b \tag{1}$$

where

i = body index
x_i = set of generalized coordinates describing the motion of body i as a free body in inertial space; this set of coordinates can be anything from geometric coordinates to free-free normal modes to cantilever modes augmented by six rigid body modes for the fixed end
M_i = generalized mass matrix for body i
K_i = generalized stiffness matrix for body i
u = set of control inputs
G_i = control distribution matrix for body i
n_b = number of bodies

System Equations of Motion

A multibody system is created by constraining the components to share certain common motions and by adding flexible connections between bodies. Assume that the constraints can be described in the following form:

$$\sum_{i=1}^{n_b} A_i x_i = 0, \qquad i = 1,\ldots, n_b$$

or

$$AX = 0 \tag{2}$$

where

$$A = [A_1 A_2 \cdots A_{n_b}], \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_b} \end{bmatrix}$$

Let n_c be the number of constraint equations in Eq. (2). The constraints may be introduced into the equations of motion using a column vector Λ of Lagrange multipliers. The constrained system is

$$M_i\ddot{x}_i + K_i x_i = G_i u + A_i^T \Lambda, \qquad i = 1,\ldots, n_b \tag{3}$$

$$A\dot{X} = 0 \tag{4}$$

Let P be any full rank matrix mapping a minimal system state x into X:

$$X = Px, \qquad \text{or} \qquad x_i = P_i x, \qquad i = 1,\ldots, n_b \tag{5}$$

The constraint equation becomes

$$AP\dot{x} = 0$$

Since the states X are independent, $AP = 0$, or

$$\sum_{i=1}^{n_b} A_i P_i = 0 \tag{6}$$

Once P is chosen so that Eq. (6) is satisfied, the constraint equation [Eq. (4)] is automatically satisfied. P may take many forms. If the constraints take the form "state 5 in body 1 has the same value as state 1 in body 2," then the system states may be selected body states and P may merely be a permutation matrix—mapping system states 1-4 into states 1-4 of body 1, system state 5 into both state 5 of body 1 and state 1 of body 2, and so forth. Although such a choice of physical system states has desirable features, in some cases it may be convenient to use system states that are not physically meaningful. For an arbitrary constraint matrix A, a singular value or other decomposition may be formed to find the null space of A. Any choice of P whose columns span the null space of A may be used to define a minimal set of system states x. While the system states may be physically meaningless, Eq. (5) provides the output relation to connect these to the physical states of the problem.

Inserting Eq. (5) into Eq. (3) and premultiplying by P_i^T gives

$$P_i^T M_i P_i \ddot{x} + P_i^T K_i P_i x = P_i^T G_i u + P_i^T A_i^T \Lambda, \qquad i = 1,\ldots, n_b \tag{7}$$

Summing over i,

$$M\ddot{x} + Kx = Gu \tag{8}$$

where

$$M = \sum_{i=1}^{n_b} P_i^T M_i P_i \tag{9}$$

$$K = \sum_{i=1}^{n_b} P_i^T K_i P_i \tag{10}$$

$$G = \sum_{i=1}^{n_b} P_i^T G_i \tag{11}$$

Equation (8) is the system equation of motion incorporating all constraints. Equation (8) may be converted to the modal form:

$$x = \Phi q \tag{12a}$$

$$\ddot{q} + \Omega^2 q = \Phi^T G u \tag{12b}$$

System Model Reduction

Suppose we choose some model-reduction method that yields as its output a set of n_R modes, q_R, to be retained with the remaining set of n_Z modes, q_Z, to be zeroed. Then, we can partition Φ and write

$$x = [\Phi_R \Phi_Z] \begin{bmatrix} q_R \\ q_Z \end{bmatrix} \tag{13}$$

By setting $q_Z = 0$, we obtain the reduced-order system model

$$\ddot{q}_R + \Omega_R^2 q_R = \Phi_R^T G u \tag{14}$$

$$x = \Phi_R q_R \tag{15}$$

The homogeneous solutions to Eq. (12b) are of the form

$$q = e_j \cos(\omega_j t)$$

where t denotes time, $\omega_j^2 = \Omega_{jj}^2$, and for the column vector e_j, the jth component is equal to 1 and all other components are equal to 0. In this equation and those following, q, x, and Λ are solutions for a particular choice of j, but for clarity, this dependency is not shown explicitly in the notation. Taking x from Eq. (13), X from Eq. (5), and limiting j to $(1 \le j \le n_R)$, we find the following set of homogeneous solutions for x_i

$$x_i = P_i \Phi_R e_j \cos(\omega_j t), \qquad i = 1,\dots, n_b, \qquad j = 1,\dots, n_R$$

differentiating twice

$$\ddot{x}_i = P_i \Phi_R e_j (-\omega_j^2) \cos(\omega_j t), \qquad i = 1,\dots, n_b, \qquad j = 1,\dots, n_R$$

Inserting the preceding two equations into each of Eq. (3) for $u = 0$, we find

$$[M_i(-\omega_j^2) + K_i] P_i \Phi_R e_j \cos(\omega_j t) = A_i^T \Lambda$$

$$i = 1,\dots, n_b, \qquad j = 1,\dots, n_R$$

Premultiplying the preceding equation by A_i and summing over i gives

$$\sum_{i=1}^{n_b} A_i [M_i(-\omega_j^2) + K_i] P_i \Phi_R e_j \cos(\omega_j t) = AA^T\Lambda, \quad j = 1,\dots, n_R$$

Solving for Λ

$$\Lambda = \left\{ [AA^T]^{-1} \sum_{i=1}^{n_b} A_i [M_i(-\omega_j^2) + K_i] P_i \Phi_R e_j \right\} \cos(\omega_j t)$$

$$j = 1,\dots, n_R$$

The term in braces is a constant depending only on j and may be given the symbol Λ_{oj}. With this definition,

$$[M_i(-\omega_j^2) + K_i] P_i \Phi_R e_j = A_i^T \Lambda_{oj}, \qquad i = 1,\dots, n_b \tag{16}$$

This relation will be needed in a later derivation.

Component Model Reduction

Although none of the material in the system model reduction section is unique to the projection and assembly method, it serves as a starting point for the development of the method. The idea on which the method is based is as follows: Cause each component to have, as an allowable motion, the mode shape of each retained mode projected onto the component. When the system is reassembled from reduced-order components, the retained mode will still be an admissible motion of the reduced-order system. In the following, it will be shown that in addition to being an admissible motion of the reduced-order system, it is a mode of the reduced-order system.

Consider the projection of q_R onto component i. Using Eqs. (5) and (15), we have

$$x_i = P_i \Phi_R q_R \tag{17}$$

In general, q_R should be of lower order than x_i. Where before, component i had n_i degrees of freedom, Eq. (15) restricts the motion to n_R degrees of freedom. Let x_{Ri} be a set of component i modes that span the space of component motions allowed by Eq. (15). In Ref. 10, the choice $x_{Ri} = q_R$ is made, so

$$x_i = P_i \Phi_R x_{Ri} \tag{18}$$

Implicit in this choice is the assumption that the matrix $P_i \Phi_R$ is of full column rank. This assumption is violated in a number of situations. The most obvious case is when one component has fewer degrees of freedom than the number of modes in Φ_R. Other examples arise when the projections of the modes are linearly dependent within the subspace of a particular component. In a later section of this paper, an alternative choice for x_{Ri} is explored for situations where $P_i \Phi_R$ is not of full rank. Writing the component equations of motion [Eq. (3)] and constraint relation [Eq. (4)] in terms of the x_{Ri},

$$M_{Ri} \ddot{x}_{Ri} + K_{Ri} x_{Ri} = G_{Ri} u + A_{Ri}^T \Lambda, \qquad i = 1,\dots, n_b \tag{19}$$

$$A_R \dot{X}_R = \sum_{i=1}^{n_b} A_{Ri} \dot{x}_{Ri} = 0 \tag{20}$$

where

$$M_{Ri} = \Phi_R^T P_i^T M_i P_i \Phi_R \tag{21}$$

$$K_{Ri} = \Phi_R^T P_i^T K_i P_i \Phi_R \tag{22}$$

$$G_{Ri} = \Phi_R^T P_i^T G_i \tag{23}$$

$$A_{Ri} = A_i P_i \Phi_R \tag{24}$$

and

$$A_R = [A_{R1} A_{R2} \cdots A_{Rn_b}], \qquad X_R = \begin{bmatrix} x_{R1} \\ x_{R2} \\ \vdots \\ x_{Rn_b} \end{bmatrix}$$

This system of equations in x_{Ri} and Λ may be formulated in terms of a minimal set of states x_R with some mapping P_R:

$$X_R = P_R x_R \tag{25}$$

with this choice, Eq. (20) becomes

$$A_R P_R x_R = 0 \tag{26}$$

Since the x_R are independent, this requires

$$A_R P_R = 0 \tag{27}$$

In actual practice, P_R has a specific form, but to understand the behavior of the reduced-order system, we can consider any P_R that is of full rank and satisfies Eq. (27). If $x_{R1} = x_{R2} = \cdots = x_{Rn_b}$ (as will be the case for the desired retained modes), then $x_{Ri} = x_{R1}$, and Eq. (20) becomes

$$\left(\sum_{i=1}^{n_b} A_i P_i \right) \Phi_R x_{R1} = 0$$

which is automatically satisfied in view of Eq. (6). This suggests that a partial choice for P_R is the column: $[I I \cdots I]^T$. A full rank P_R that satisfies Eq. (20) may be created by taking the singular value decomposition of a portion of A_R:

$$[A_{R2} \cdots A_{Rn_b}] = U_A \Sigma_A V_R^T = U_A [\Sigma_{A1} \; 0] \begin{bmatrix} V_{A1}^T \\ V_{A2}^T \end{bmatrix} = U_A \Sigma_{A1} V_{A1}^T \tag{28}$$

and choosing

$$P_R^T = \begin{bmatrix} I & I \cdots I \\ 0 & V_{A2}^T \end{bmatrix} \tag{29}$$

so

$$A_R P_R = \left[\left(\sum_{i=1}^{n_b} A_{Ri} \right) \quad [A_{R2} \cdots A_{Rn_b}] V_{A2} \right]$$

$$= \left[\left(\sum_{i=1}^{n_b} A_i P_i \right) \Phi_R \quad U_A \Sigma_{A1} V_{A1}^T V_{A2} \right]$$

$$= [0 \; 0] = 0$$

as desired. Furthermore, P_R is of full rank by construction. This choice for P_R is used here only to simplify the derivations that follow. It is not recommended for use in actual computation. For example, the first two examples in this paper redefine U_A, Σ_A, and V_A^T by taking the decomposition of the entire A_R matrix rather than just a portion as before; that is

$$A_R = U_A \Sigma_A \begin{bmatrix} V_{A1}^T \\ V_{A2}^T \end{bmatrix}$$

and then

$$P_R = V_{A2}$$

Starting from Eq. (19), the equations of motion in terms of x_R are

$$\sum_{i=1}^{n_b} P_{Ri}^T M_{Ri} P_{Ri} \ddot{x}_R + \sum_{i=1}^{n_b} P_{Ri}^T K_{Ri} P_{Ri} x_R = \sum_{i=1}^{n_b} P_{Ri}^T G_{Ri} u + P_R^T A_R^T \Lambda \tag{30}$$

The form of P_R in Eq. (29) suggests a partitioning of x_R and P_R into desired and extra states:

$$x_R = \begin{bmatrix} x_D \\ x_E \end{bmatrix}, \qquad P_R = [P_{RD}\ P_{RE}]$$

where

$$P_{RDi} = I \text{ and } P_{RE} = \begin{bmatrix} 0 \\ V_{A2} \end{bmatrix}$$

In partitioned form, Eq. (30) is

$$\begin{bmatrix} \sum_{i=1}^{n_b} M_{Ri} & \sum_{i=1}^{n_b} M_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} & \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} \ddot{x}_D \\ \ddot{x}_E \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n_b} K_{Ri} & \sum_{i=1}^{n_b} K_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} & \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} x_D \\ x_E \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_b} G_{Ri} \\ \sum_{i=1}^{n_b} P_{REi}^T G_{Ri} \end{bmatrix} u \tag{31}$$

By construction, the system is capable of taking the shape of any of the n_R desired modes. It remains to be shown that the x_D are free-free normal modes of the reduced-order system. To show that they are requires only that

$$x_R = \begin{bmatrix} e_j \\ 0 \end{bmatrix} \cos\omega_j t$$

be a solution of Eq. (31) with $u = 0$. Assume that

$$x_R = \begin{bmatrix} e_j \\ 0 \end{bmatrix} \cos\omega_j t$$

then

$$\ddot{x}_R = (-\omega_j^2) \begin{bmatrix} e_j \\ 0 \end{bmatrix} \cos\omega_j t$$

and

$$\left[\sum_{i=1}^{n_b} M_{Ri}(-\omega_j^2) + \sum_{i=1}^{n_b} K_{Ri} \right] e_j \cos\omega_j t \overset{?}{=} 0 \tag{32}$$

$$\left[\sum_{i=1}^{n_b} P_{REi}^T M_{Ri}(-\omega_j^2) + \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} \right] e_j \cos\omega_j t \overset{?}{=} 0 \tag{33}$$

If both left-hand sides in the preceding equations evaluate to zero, then the desired modes are modes of the reduced-order system. In the following derivations, a number in parentheses under an equals sign is the number of an earlier equation used in the derivation. Consider ΣM_{Ri} and ΣK_{Ri}:

$$\sum_{i=1}^{n_b} M_{Ri} \underset{(21)}{=} \sum_{i=1}^{n_b} \Phi_R^T P_i^T M_i P_i \Phi_R = \Phi_R^T \sum_{i=1}^{n_b} P_i^T M_i P_i \Phi_R$$

$$\underset{(9)}{=} \Phi_R^T M \Phi_R = I$$

Similarly,

$$\sum_{i=1}^{n_b} K_{Ri} = \Omega^2$$

Eq. (32) becomes

$$[(-\omega_j^2)I + \Omega^2]e_j = \begin{Bmatrix} (\omega_i^2 - \omega_j^2)(0), & i \neq j \\ (\omega_i^2 - \omega_j^2)(1), & i = j \end{Bmatrix} = 0$$

and so is satisfied. Consider Eq. (33):

$$\left[\sum_{i=1}^{n_b} P_{REi}^T M_{Ri}(-\omega_j^2) + \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} \right] e_j$$

$$\underset{(21,22)}{=} \left\{ \sum_{i=1}^{n_b} P_{REi}^T \Phi_R^T P_i^T [M_i(-\omega_j^2) + K_i] P_i \Phi_R \right\} e_j$$

$$\underset{(16)}{=} \sum_{i=1}^{n_b} P_{REi}^T \Phi_R^T P_i^T A_i^T \Lambda_{oj}$$

$$\underset{(24)}{=} \sum_{i=1}^{n_b} P_{REi}^T A_{Ri}^T \Lambda_{oj}$$

$$= [0\ V_{A2}^T] A_R^T \Lambda_{oj}$$

$$\underset{(28)}{=} 0 \Lambda_{oj} = 0$$

Therefore, the desired mode shapes and frequencies satisfy Eq. (31) and thus are normal modes of the reassembled reduced-order system.

Component Model Reduction—Extended Method

As previously mentioned, the choice $x_{Ri} = q_R$ depends on the matrix $P_i \Phi_R$ being of full column rank. When this is not the case, the method can be extended to allow model reduction to proceed. Consider the singular-value decomposition (SVD) of $P_i \Phi_R$. Suppressing the index i on the products in the SVD, let r be the rank of $P_i \Phi_R$, let n_R be the rank of Φ_R, and let n_i be the rank of P_i (and the number of states in x_i). If $r = n_R < n_i$ (Ref. 10),

$$P_i \Phi_R = U \Sigma V^T = [U_1\ U_2] \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} V^T = U_1 \Sigma_1 V^T \tag{34}$$

If $r = n_i < n_R$ (body i has few DOF),

$$P_i \Phi_R = U \Sigma V^T = U [\Sigma_1\ 0] \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U \Sigma_1 V_1^T \tag{35}$$

If $r < n_i$, $r < n_R$ (linear-dependent projected modes),

$$P_i \Phi_R = U \Sigma V^T = [U_1\ U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T \tag{36}$$

Choose $x_{Ri} = S_i q_R$. To ensure that the set x_{Ri} is an independent set spanning the space of component motions, choose $S_i = \Sigma_1(i)V_1^T(i)$. The method just described used $S_i = I$. In the event that $r(i) = n_R$, $V_1^T(i)$ becomes $V(i)$. This choice of x_{Ri} gives, for Eq. (18),

$$x_i = U_1(i)x_{Ri} \tag{37}$$

In the event that $r(i) = n_i$, $U_1(i)$ becomes $U(i)$. Define $Q_i = U_1(i)$. Equations (21-24) now take the form:

$$M_{Ri} = Q_i^T M_i Q_i \tag{38a}$$

$$K_{Ri} = Q_i^T K_i Q_i \tag{38b}$$

$$G_{Ri} = Q_i^T G_i \tag{38c}$$

$$A_{Ri} = A_i Q_i \tag{38d}$$

Choosing

$$P_R^T = \begin{bmatrix} S_1^T & [S_2^T \cdots S_{n_b}^T] \\ 0 & V_{A2}^T \end{bmatrix} \tag{39}$$

gives as desired

$$A_R P_R = 0 \tag{40}$$

Moreover, P_R is again full rank by construction. Partition P_R as before:

$$P_R = [P_{RD} \; P_{RE}]$$

where

$$P_{RDi} = S_i \text{ and } P_{RE} = \begin{bmatrix} 0 \\ V_{A2} \end{bmatrix}$$

Equation (31) becomes

$$\begin{bmatrix} \sum_{i=1}^{n_b} P_{RDi}^T M_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{RDi}^T M_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{REi}^T M_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} \ddot{x}_D \\ \ddot{x}_E \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n_b} P_{RDi}^T K_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{RDi}^T K_{Ri} P_{REi} \\ \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} P_{RDi} & \sum_{i=1}^{n_b} P_{REi}^T K_{Ri} P_{REi} \end{bmatrix} \begin{bmatrix} x_D \\ x_E \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_b} P_{RDi}^T G_{Ri} \\ \sum_{i=1}^{n_b} P_{REi}^T G_{Ri} \end{bmatrix} u \tag{41}$$

and the proof that the desired modes are normal modes of the reassembled reduced-order system proceeds exactly as before, using the preceding definition of P_{RDi} and Eq. (38).

Simple Examples

One-Dimensional, Three-Disk Example

Consider Fig. 1. In this example, there are three disks with rotational displacements (from left to right) y_1, y_2, and y_3 and inertias $4J$, J, and J connected to ground and each other by torsion rods of equal spring constant k. We choose to consider this simple system as being composed of two simpler subsystems of components. We divide the middle disk in half and allocate one half to each subsystem. Subsystem 1 contains the large disk and the left half of the middle disk. Take $x_1 = [y_1, y_2]^T$. Subsystem 2 contains the rest of the system. Take $x_2 = [y_2, y_3]^T$. Choosing units to make J and k equal to unity, the mass and stiffness matrices for each component are

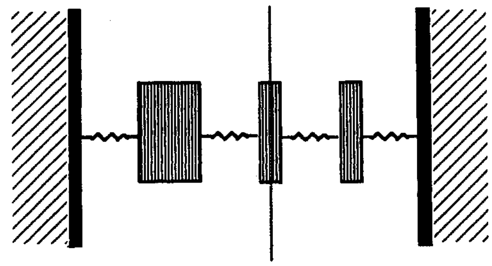

Fig. 1 One-dimensional, three-disk example.

$$M_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

The constraint relation that connects the subsystems is that $x_1(2) = x_2(1)$. Express in terms of a constraint matrix A:

$$A = [A_1 \; A_2], \qquad [A_1 \; A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

with

$$A_1 = [0 \quad 1], \qquad A_2 = [-1 \quad 0]$$

One choice of P that reduces this to minimal form is

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that for such a simple constraint relationship, a minimal set of physically meaningful coordinates may be easily identified without resorting to SVD routines. An example would be the choice of $P = P'$:

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The SVD approach is demonstrated here to aid in the process of generalizing to more complex systems.

The system mass and stiffness matrices are

$$M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 0.7071 & 0 \\ 0.7071 & 1 & -0.7071 \\ 0 & -0.7071 & 2 \end{bmatrix}$$

The eigenvalue and eigenvector matrices for this system are

$$\Omega^2 = \begin{bmatrix} 0.2803 & 0 & 0 \\ 0 & 1.1694 & 0 \\ 0 & 0 & 3.0502 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0.4457 & -0.2153 & 0.0703 \\ -0.5539 & -0.8155 & 1.0140 \\ -0.2277 & -0.6943 & -0.6827 \end{bmatrix}$$

System Model Reduction

Assume we wish to capture only the lowest frequency system mode ($\Omega^2 = 0.2803$), then

$$\Phi_R = \begin{bmatrix} 0.4457 \\ -0.5539 \\ -0.2277 \end{bmatrix}$$

Component Model Reduction

Choose $x_{R1} = x_{R2} = q_R$, so that

$$x_1 = P_1 \Phi_R x_{R1} = \begin{bmatrix} -0.4457 \\ -0.3917 \end{bmatrix} x_{R1}$$

$$x_2 = P_2 \Phi_R x_{R2} = \begin{bmatrix} -0.3917 \\ -0.2277 \end{bmatrix} x_{R2}$$

and the reduced-order component mass and stiffness matrices are

$$M_{R1} = [0.8714], \qquad K_{R1} = [0.2016]$$

$$M_{R2} = [0.1286], \qquad K_{R2} = [0.0787]$$

The reduced-order constraint matrix is

$$A_R = [-0.3916 \quad 0.3916]$$

Choose P_R:

$$P_R = \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix}$$

This gives the reduced-order system

$$[0.5]\ddot{x}_R + [0.1402]x_R = 0$$

which has a single eigenvalue at $\Omega^2 = 0.2803$. In this case, no extra modes are created because it happens that ($2n_R$ − number of constraints) $= n_R$. This is not true in general. The next example produces extra modes.

One-Dimensional, Five-Disk Example

Consider Fig. 2. In this example, there are five disks with displacements (from left to right) y_1, y_2, y_3, y_4, and y_5 and inertias $4J$, J, J, J, and J connected to ground and each other by torsion rods of equal spring constant k. We choose to consider this simple system as being composed of two simpler subsystems of components. We divide the middle disk in half and allocate one half to each subsystem. Subsystem 1 contains the large disk through the left half of the middle disk. Take $x_1 = [y_1, y_2, y_3]^T$. Subsystem 2 contains the rest of the system. Take $x_2 = [y_3, y_4, y_5]^T$. Choosing units to make J and k equal to unity, the mass and stiffness matrices for each component are

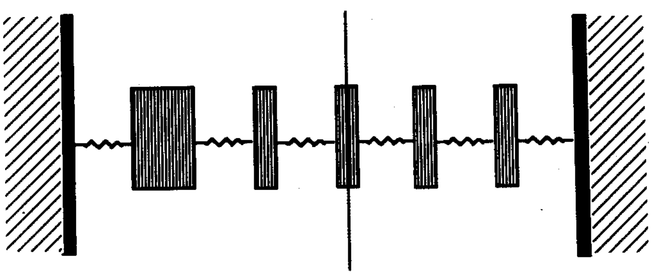

Fig. 2 One-dimensional, five-disk example.

$$M_1 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The constraint relation that connects the subsystems is that $x_1(3) = x_2(1)$. Expressed in terms of a constraint matrix A:

$$A = [A_1 \; A_2], \qquad [A_1 \; A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

with

$$A_1 = [0 \quad 0 \quad 1], \qquad A_2 = [-1 \quad 0 \quad 0]$$

One choice of P that reduces this to minimal form is

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & 0.7071 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The system mass and stiffness matrices are

$$M = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 0.75 & -0.25 & 0 & 0 \\ 0 & -0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & -0.7071 & 0.7071 & 0 & 0 \\ -0.7071 & 2.2071 & -0.5 & -0.5 & 0 \\ 0.7071 & -0.5 & 0.7929 & -0.5 & 0 \\ 0 & -0.5 & -0.5 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

The eigenvalue and eigenvector matrices for this system are

$$\Omega^2 = \begin{bmatrix} 0.1933 & 0 & 0 & 0 & 0 \\ 0 & 0.5466 & 0 & 0 & 0 \\ 0 & 0 & 1.4696 & 0 & 0 \\ 0 & 0 & 0 & 2.6609 & 0 \\ 0 & 0 & 0 & 0 & 3.6296 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} -0.3489 & 0.3208 & -0.1392 & -0.0704 & -0.0310 \\ 0.1217 & 0.3655 & -0.0438 & 0.7626 & 0.8765 \\ 0.7270 & 0.4501 & -0.8075 & -0.0986 & 0.3272 \\ 0.3386 & 0.5329 & 0.3142 & 0.3895 & -0.5924 \\ 0.1874 & 0.3666 & 0.5924 & -0.5894 & 0.3635 \end{bmatrix}$$

System Model Reduction

Assume we wish to capture only the two lowest frequency system modes ($\Omega^2 = 0.1933, 0.5466$), then

$$\Phi_R = \begin{bmatrix} -0.3489 & 0.3208 \\ 0.1217 & 0.3655 \\ 0.7270 & 0.4501 \\ 0.3386 & 0.5329 \\ 0.1874 & 0.3666 \end{bmatrix}$$

Component Model Reduction

Choose $x_{R1} = x_{R2} = q_R$, so

$$x_1 = P_1\Phi_R x_{R1} = \begin{bmatrix} 0.3489 & -0.3208 \\ 0.4280 & 0.0598 \\ 0.4244 & 0.4078 \end{bmatrix} x_{R1}$$

$$x_2 = P_2\Phi_R x_{R2} = \begin{bmatrix} 0.4244 & 0.4078 \\ 0.3386 & 0.5329 \\ 0.1874 & 0.3666 \end{bmatrix} x_{R2}$$

and the reduced-order component mass and stiffness matrices are

$$M_{R1} = \begin{bmatrix} 0.7602 & -0.3357 \\ -0.3357 & 0.4985 \end{bmatrix}, \quad K_{R1} = \begin{bmatrix} 0.1280 & -0.0831 \\ -0.0831 & 0.3689 \end{bmatrix}$$

$$M_{R2} = \begin{bmatrix} 0.2398 & 0.3357 \\ 0.3357 & 0.5015 \end{bmatrix}, \qquad K_{R2} = \begin{bmatrix} 0.0653 & 0.0831 \\ 0.0831 & 0.1777 \end{bmatrix}$$

The reduced-order constraint matrix is

$$A_R = [0.4243 \quad 0.4078 \quad -0.4243 \quad -0.4078]$$

Choose P_R

$$P_R = \begin{bmatrix} -0.8603 & 0 & 0 \\ 0.2904 & 0.5926 & 0.5696 \\ -0.3021 & 0.7762 & -0.2151 \\ -0.2904 & -0.2151 & 0.7933 \end{bmatrix}$$

This gives the reduced-order system

$$\begin{bmatrix} 0.8954 & 0.1781 & 0.0875 \\ 0.1781 & 0.2307 & 0.2649 \\ 0.0875 & 0.2649 & 0.3739 \end{bmatrix} \ddot{x}_R + \begin{bmatrix} 0.2029 & 0.0883 & 0.0503 \\ 0.0883 & 0.1494 & 0.1383 \\ 0.0503 & 0.1383 & 0.2062 \end{bmatrix} x_R = 0$$

which has three eigenvalues at $\Omega^2 = (0.1933, 0.5466, 1.6873)$. The first two are the desired system modes, whereas the third does not match any of the original system modes; it is an "extraneous mode."

Large and Small Subsystem Example

Consider Fig. 3. This is the same system as in example 2, but with a different division into subsystems. Subsystem 1 contains the large disk through the left half of the right-most disk. Take $x_1 = [y_1, y_2, y_3, y_4, y_5]^T$. Subsystem 2 contains the rest of the right-most disk. Take $x_2 = [y_5]^T$. Choosing units to make J and k equal to unity, the mass and stiffness matrices for each component are

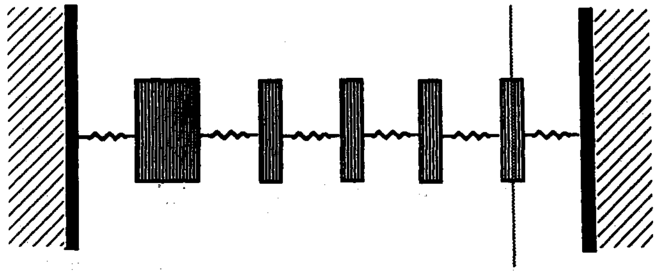

Fig. 3 Large and small subsystem example.

$$M_1 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, \qquad M_2 = [0.5]$$

$$K_1 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \qquad K_2 = [1]$$

The constraint relation that connects the subsystems is that $x_1(5) = x_2(1)$. Expressed in terms of a constraint matrix A:

$$A = [A_1 \; A_2], \qquad [A_1 \; A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

with

$$A_1 = [0 \quad 0 \quad 0 \quad 0 \quad 1], \qquad A_2 = [-1]$$

One choice of P that reduces this to minimal form is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The system mass and stiffness matrices are

$$M = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

The eigenvalue and eigenvector matrices for this system are

$$\Omega^2 = \begin{bmatrix} 0.1933 & 0 & 0 & 0 & 0 \\ 0 & 0.5466 & 0 & 0 & 0 \\ 0 & 0 & 1.4696 & 0 & 0 \\ 0 & 0 & 0 & 2.6609 & 0 \\ 0 & 0 & 0 & 0 & 3.6296 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0.8152 & -0.6021 & 0.2351 & -0.1157 & 0.0515 \\ 1.0000 & 0.1123 & -0.9116 & 1.0000 & -0.6453 \\ 0.9914 & 0.7653 & -0.7186 & -0.5452 & 1.0000 \\ 0.7911 & 1.0000 & 0.5304 & -0.6397 & -0.9843 \\ 0.4379 & 0.6881 & 1.0000 & 0.9679 & 0.6040 \end{bmatrix}$$

System Model Reduction

Assume we wish to capture only the two lowest frequency system modes ($\Omega^2 = 0.1933, 0.5466$), then

$$\Phi = \begin{bmatrix} 0.8152 & -0.6021 \\ 1.0000 & 0.1123 \\ 0.9914 & 0.7653 \\ 0.7911 & 1.0000 \\ 0.4379 & 0.6881 \end{bmatrix}$$

Component Model Reduction

Following the extended method, consider $P_i\Phi_R$. Note that Φ_R is of rank 2: $n_R = 2$. P_1 is the 5×5 identity matrix and P_2 is 1×5, the last row of P. So $P_1\Phi_R = \Phi_R$, which has full column rank by definition. However, $P_2\Phi_R = [\Phi_{R51} \ \Phi_{R52}]$, which has two columns but only one row and so is not of full column rank; it has only rank $r = 1$. In the extended method, the choice $x_{Ri} = S_i q_R$ is made. For component 1, S_1 can be chosen as the 2×2 identity matrix, giving $x_{R1} = q_R$, $Q_1 = P_1\Phi_R$ as before. For component 2, start by taking the singular-value decomposition of $P_2\Phi_R$. In this simple example, this can be done by inspection:

$$P_2\Phi_R = [1][|\Phi_{R5}|]\left[\frac{\Phi_{R51}}{|\Phi_{R5}|} \ \frac{\Phi_{R52}}{|\Phi_{R5}|}\right]$$

and $S_2 = [0.4379 \ 0.6881]$ and $Q_2 = [1]$. $Q_2 = 1$ means that the reduced component 2 system of equations will be identical to the original component 2 equations. Since there are fewer modes in component 2 than the desired reduced-order system already, component 2 does not have to change. In particular,

$$x_{R2} = x_2, \quad M_{R2} = M_2, \quad K_{R2} = K_2, \quad A_{R2} = A_2$$

For component 1,

$$x_1 = P_1\Phi_R x_{R1} = \begin{bmatrix} 0.8152 & -0.6021 \\ 1.0000 & 0.1123 \\ 0.9914 & 0.7653 \\ 0.7911 & 1.0000 \\ 0.4379 & 0.6881 \end{bmatrix} x_{R1}$$

and the reduced-order component mass and stiffness matrices are

$$M_{R1} = \begin{bmatrix} 5.3632 & -0.1506 \\ -0.1506 & 3.2852 \end{bmatrix}$$

$$K_{R1} = \begin{bmatrix} 0.8637 & -0.3013 \\ -0.3013 & 1.4517 \end{bmatrix}$$

The reduced-order constraint matrix is

$$A_R = [\Phi_{R51} \quad \Phi_{R52} \quad -1] = [0.4379 \quad 0.6881 \quad -1]$$

From Eq. (39), we can choose P_R as

$$P_R = \begin{bmatrix} S_1 & 0 \\ S_2 & V_{A2} \end{bmatrix}$$

where V_{A2} is interesting in this example: From Eq. (28), V_{A2} comes from the SVD of a portion of A_R, in this case A_{R2}. But $A_{R2} = -1$, so

$$A_{R2} = [-1] \, [1] \, [1]$$

and V_{A1} is square, and V_{A2} has dimensions 1×0, allowing the preceding expression for P_R to be reduced to

$$P_R = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.4379 & 0.6881 \end{bmatrix}$$

This gives the reduced-order system

$$\begin{bmatrix} 5.4591 & 0 \\ 0 & 3.5219 \end{bmatrix} \ddot{x}_R + \begin{bmatrix} 1.0555 & 0 \\ 0 & 1.9252 \end{bmatrix} x_R = 0$$

which has two eigenvalues at $\Omega^2 = (0.1933, 0.5466)$. These are the desired system modes.

Conclusions

In this paper, the projection and assembly model-reduction method has been analyzed to demonstrate why the desired modes are returned exactly. An explicit set of necessary conditions involving the rank of the projection matrix has been presented, and an extension to the method has been proposed that removes those conditions. The method was demonstrated using simple examples.

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